Tail asymptotics of light-tailed Weibull-like sums

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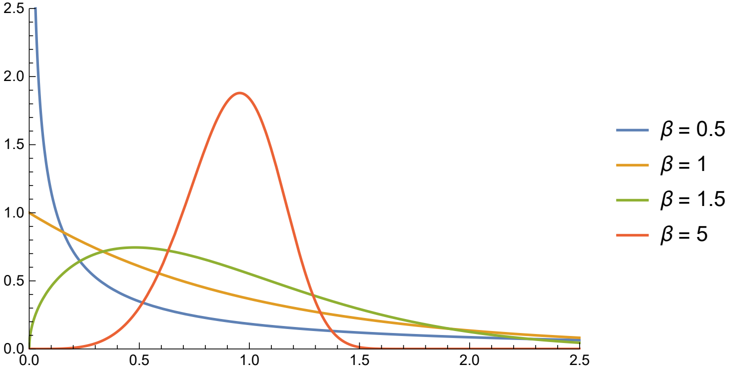
Weibull distribution

For $X \sim \mathrm{Weibull}(\beta)$ we have pdf $f(x) = \beta x^{\beta-1} \mathrm{e}^{-x^\beta}$ and tail $\overline{F}(x) = \mathrm{e}^{-x^\beta}$.



(Inspired by Wikipedia)

Popular due to Fisher–Tippet theorem. It can be *both light-tailed and heavy-tailed!*
We want

$$\mathbb{P}(X_1 + \cdots + X_n > x) \sim \mathrm{SomeAsymptoticForm}(x; \beta) \text{ as } x \to \infty$$

for **iid** case, **fixed** n.

No-one has really looked at Weibull sums.

A Review of Results on Sums of Random Variables

Saralees Nadarajah

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6 Sums of Weibull RVs

Unfortunately, no results (not even approximations) have been known for sums of Weibull random variables. It is expected that this review could help to motivate some work for this case.

Asymptotics you guess

For $\beta < 1$ then X is *heavy-tailed* so *principle of the single big jump*

$$\mathbb{P}(X_1 + \cdots + X_n > x) \sim d\overline{F}(x) \,.$$

For $\beta = 1$ then $X \sim \mathrm{Exponential}(1)$ so $\sum_{i=1}^n X_i \sim \mathrm{Erlang}(n, 1)$, so $\overline{F}(x)$ known.

For $\beta > 1$ then X is *light-tailed* so we might expect

$$\mathbb{P}(X_1 + \cdots + X_n > x) \overset{?}{\sim} \overline{F}\left(\frac{x}{n}\right)^n \,.$$

Asymptotics known but hard to find

DENSITIES WITH GAUSSIAN TAILS

A. A. BALKEMA, C. KLÜPPELBERG, *and* S. I. RESNICK

[Received 29 April 1991—Revised 16 March 1992 and 16 April 1992]

Abstract

Consider densities $f_i(t)$, for $i = 1, \ldots, d$, on the real line which have thin tails in the sense that, for each i,

$$f_i(t) \sim \gamma_i(t) e^{-\psi_i(t)},$$

where γ_i behaves roughly like a constant and ψ_i is convex, C^2, with $\psi' \to \infty$ and $\psi'' > 0$ and $1/\sqrt{\psi''}$ is self-neglecting. (The latter is an asymptotic variation condition.) Then the convolution is of the same form

$$f_1 * \ldots * f_d(t) \sim \gamma(t) e^{-\psi(t)}.$$

Formulae for γ, ψ are given in terms of the factor densities and involve the conjugate transform and infimal convolution of convexity theory. The derivations require embedding densities in exponential families and showing that the assumed form of the densities implies asymptotic normality of the exponential families.

Our paper

TAIL ASYMPTOTICS OF LIGHT-TAILED WEIBULL-LIKE SUMS

BY

SØREN ASMUSSEN (AARHUS), ENKELEJD HASHORVA (LAUSANNE), PATRICK J. LAUB (BRISBANE) AND THOMAS TAIMRE (BRISBANE)

Abstract. We consider sums of n i.i.d. random variables with tails close to $\exp\{-x^\beta\}$ for some $\beta > 1$. Asymptotics developed by Rootzén (1987) and Balkema, Klüppelberg & Resnick (1993) are discussed from the point of view of tails rather of densities, using a somewhat different angle, and supplemented with bounds, results on a random number N of terms, and simulation algorithms.

Submitted to *Probability and Mathematical Statistics*, a special issue for Tomasz Rolski.

How is this different to BKR?

We:

- have a better title & say "Weibull" a lot!
- have simpler proofs.
- add new results:
  - look at mgf asymptotics.
  - simulate exponentially-tilted Weibulls.
  - have log-asymptotics for the Poisson random sum of Weibulls.

Results (very briefly) I

Corollary

Say X has pdf $f(x) \sim dx^{\alpha+\beta-1}\mathrm{e}^{-cx^{\beta}}$ as $x \to \infty$. Then the tail and the density of an i.i.d. sum satisfy

$$\overline{F^{*n}}(x) = \mathbb{P}(S_n > x) \sim k(n)x^{\alpha(n)}\mathrm{e}^{-c(n)x^{\beta}}, \tag{1}$$

$$f^{*n}(x) \sim \beta c(n)k(n)x^{\alpha(n)+\beta-1}\mathrm{e}^{-c(n)x^{\beta}} \tag{2}$$

where $c(n) = c/n^{\beta-1}$, $\alpha(n) = n\alpha + (n-1)\beta/2$ and

$$k(n) = \frac{d^n}{\beta c}\left[\frac{2\pi}{\beta(\beta-1)c}\right]^{(n-1)/2} n^{\frac{1}{2}(\beta-n(2\alpha+\beta)-1)}. \tag{3}$$

Results (very briefly) II

Proposition

Say X has a pdf $f(x) \sim \gamma(x)\mathrm{e}^{-\psi(x)}$ where the function ψ is non-negative, convex, C^2, and its first order derivative is denoted λ.

As $x \to \infty$, it holds that

$$\widehat{F}[\lambda(x)] \sim \sqrt{\frac{2\pi}{\lambda'(x)}}\gamma(x)\mathrm{e}^{(\beta-1)x^\beta}, \tag{4}$$

$$\mathbb{E}_{\lambda(x)}X \sim x. \tag{5}$$

Further, we have the following convergence in $\mathbb{P}_{\lambda(x)}$-distribution as $x \to \infty$

$$\sqrt{\lambda'(x)}(X-x) = \sqrt{\beta(\beta-1)x^{\beta-2}}(X-x) \xrightarrow{\mathcal{D}} N(0,1)\,. \tag{6}$$

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Conclusion

In conclusion, check out our paper at `http://math.au.dk/publs?publid=1099`.

Corollary (Life after PhD)

I submit my thesis in January 2018 and need a "job" then.

Interests: applied probability, Monte Carlo, finance, computational/experimental maths, machine learning, statistics, big data, economics, artificial intelligence, computer science, history, international relations, public health, genetics, books, real world problems...

Locations: Melbourne, Sydney, Switzerland, Germany, Austria, Canada, France, Spain, UK, Japan, pretty much anywhere that's not North Korea.

Supported by: ACEMS, University of Queensland, Aarhus University.

Thanks for listening & Happy Birthday Phil!